The complex structure of $\mathbb{C}^n$ is given by the subbundle of (1,0)-vectors:

$$T^{1,0}_z \mathbb{C}^n = \left\{ \sum_{j=1}^n a_j \frac{\partial}{\partial z_j} : a_j \in \mathbb{C} \right\}.$$

We let $T^{0,1}_z \mathbb{C}^n = \left\{ \sum b_j \frac{\partial}{\partial \bar{z}_j} \right\} = \overline{T^{1,0}_z \mathbb{C}^n}$.

Thus, a function $f: \Omega \to \mathbb{C}$ is holom. $\Leftrightarrow$ $\bar{Z} f = 0$ for every (0,1)-vector field $\bar{Z}$ (section of $T^{0,1}\mathbb{C}^n$) over Ω (or just locally near each $z \in \Omega$).

A mapping $H = (H_1, \dots, H_N) : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^N$ is <u>holom.</u> if all components H_j are holom. It is a <u>biholomorphism</u> if $N = n$, and it has a holom. inverse $H^{-1}: \Omega' = H(\Omega) \to \Omega$. Similar to 1D, H is a local biholomorphism (change of coordinates) $\Leftrightarrow$ $\det\left(\frac{\partial H_i}{\partial z_j}\right)_{i,j} \neq 0$.

Chain rule for tangent vectors.

If $H:\Omega\to\mathbb{C}^N$ is C^k-map $(k\geq 1)$, $w=H(z)$, then

$$\begin{cases} H_*\left(\dfrac{\partial}{\partial z_i}\right)=\displaystyle\sum_{j=1}^{n}\left(\dfrac{\partial H_j}{\partial z_i}\dfrac{\partial}{\partial w_j}+\dfrac{\partial \bar{H}_j}{\partial z_i}\dfrac{\partial}{\partial \bar{w}_j}\right) \\ H_*\left(\dfrac{\partial}{\partial \bar{z}_i}\right)=\displaystyle\sum_{j=1}^{n}\left(\dfrac{\partial H_j}{\partial \bar{z}_i}\dfrac{\partial}{\partial w_j}+\dfrac{\partial \bar{H}_j}{\partial \bar{z}_i}\dfrac{\partial}{\partial \bar{w}_j}\right)\end{cases}$$

Thus, we note:

<u>Prop1.</u> A C^k-map $H:\Omega\to\mathbb{C}^N$ is <u>holom.</u>

$$\Longleftrightarrow$$

$$H_*(T^{1,0}\mathbb{C}^n)\subseteq T^{1,0}\mathbb{C}^N.$$

A consequence of this discussion is

<u>Thm1.</u> $T^{1,0}\mathbb{C}^n$ is a rank n subbundle of $\mathbb{C}T\mathbb{C}^n$ that is biholomorphically invariant, i.e., if $w=H(z)$ is a local holom. change of coordinates (a biholomorphism), then

$$H_*T^{1,0}_p\mathbb{C}^n=T^{1,0}_{H(p)}\mathbb{C}^n.$$

CR Manifolds

M^{2n-d}, $\mathbb{C}^n \cong \mathbb{R}^{2n}$

A real submanifold of $\operatorname{codim}_{\mathbb{R}} = d$, $M \subseteq \mathbb{C}^n$ is locally defined by d real functions $\rho_1, \dots, \rho_d$ w/ linearly independent ($/\mathbb{R}$) differentials: $d\rho_1 \wedge \dots \wedge d\rho_d \neq 0$ on M. Its real tangent space at p is

$$T_pM = \Big\{ X = \sum_{j=1}^{n} \Big(\alpha_j \frac{\partial}{\partial x_j} + \beta_j \frac{\partial}{\partial y_j}\Big) : \alpha_j, \beta_j \in \mathbb{R},$$
$$\forall k = 1,\dots,d : \sum_{j=1}^{n} \Big(\frac{\partial \rho_k}{\partial x_j}(p)\alpha_j + \frac{\partial \rho_k}{\partial y_j}(p)\beta_j\Big) = 0 \Big\}$$

The complexified tangent space is

$$\mathbb{C}T_pM = \mathbb{C} \otimes T_pM = \Big\{ Z = \sum_{j=1}^{n} \Big(a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j}\Big) :$$
$$a_j, b_j \in \mathbb{C}, \forall k : \sum_{j=1}^{n} \Big(\frac{\partial \rho_k}{\partial z_j}(p)\, a_j + \frac{\partial \rho_k}{\partial \bar{z}_j}(p)\, b_j\Big) = 0 \Big\}.$$

They form $\overbrace{\text{subbundles}}^{\text{of appropriate dimension}}$ of the tangent bundles $T\mathbb{C}^n$ and $\mathbb{C}T\mathbb{C}^n$, respectively.

Def. The (1,0)-tangent space $T_p^{1,0}M$ of M at $p \in M$ is

$$T_p^{1,0}M = T_p^{1,0}\mathbb{C}^n \cap \mathbb{C}T_pM = \{Z = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j} : \sum_{j=1}^n \frac{\partial \rho_k}{\partial z_j}(p) Z^j = 0,\ k = 1, \dots, d\}.$$

- Linear algebra $\Rightarrow$ $\dim_{\mathbb{C}} T_p^{1,0} = n - d'$, where $d' = \operatorname{rank}_{\mathbb{C}}(\partial\rho_1(p), \dots, \partial\rho_d(p)) \le d$.

Note. ① $d\rho_1, \dots, d\rho_d$ are lin. indep. / $\mathbb{R}$ but $\partial\rho_1, \dots, \partial\rho_d$ need not be lin. indep. / $\mathbb{C}$.

② If $d = 1$, then $d\rho \neq 0 \Leftrightarrow \partial\rho \neq 0$. Thus, $\dim_{\mathbb{C}} T_p^{1,0} M = n-1$ for all $p \in M$. Real submflds of $\operatorname{codim}_{\mathbb{R}} = d = 1$ are called real hypersurfaces.

- Ex1. Consider the complex hyperplane $M = \{z_n = 0\} \cong \mathbb{C}^{n-1}$. As a real submfld, it is defined by $\operatorname{Re} z_n = x_n = 0$ and $\operatorname{Im} z_n = y_n = 0$